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# On Asymptotic Equivalence between Two Nonlinear Parametric Systems with a Small Parameter

MEHRAN BASTI\*

*Department of Mathematics,  
University of Alberta, Edmonton T6G 2G1, Canada*

*Submitted by V. Lakshmikantham*

## INTRODUCTION

The problem of asymptotic equivalence between two systems of ordinary differential equations has been considered by many authors. One technique that is often employed in connection with this problem is to utilize a variation of constants formula in conjunction with a fixed-point theorem. The articles [5, 13, 15, 17, 20, 21] treat linear perturbation problems using this approach, while the papers [2, 6–8] employ a variant of the classical variation of constants formula introduced by Alekseev [1] as a tool for discussing nonlinear perturbation problems. Other procedures have also been employed in connection with this problem; for example N. Onuchic [19] used a topological method of Wazewski to discuss this problem in the case when the unperturbed system is linear, and in [14] the concept of admissibility is used in conjunction with the Schauder–Tychonoff fixed-point theorem. The comparison principle has also been used coupled with fixed-point theorems; see [5, 12, 15, 17, 20]. Lists of papers on this subject may be found in [2–4, 12, 22, 25–27] and also in a number of standard texts on ordinary differential equations, e.g., Coddington and Levinson [8], Hartman [16] and Sansone and Conti [24].

The purpose of this paper is to investigate these problems further. We are mainly interested in establishing asymptotic relationships between the solutions of systems

$$\dot{x} = f(t, x, \lambda, \varepsilon) \quad \cdot = \frac{d}{dt} \quad (1)$$

and

$$\dot{y} = f(t, y, \psi(t, \varepsilon), \varepsilon) + g(t, y, \varepsilon). \quad (2)$$

\* Current address: Department of Mathematics, University of Regina, Regina S4S 0A2, Canada.

Here  $f$  and the Jacobian matrices  $f_x, f_\lambda$  are continuous for  $(t, x, \lambda, \varepsilon)$  in  $R^+ \times R^n \times S_c \times (0, \varepsilon_1]$  into  $R^n$ .  $S_c$  is the closed ball of radius  $c$  in  $R^m$ .  $m$  and  $n$  are positive integers.  $R^+ = [0, \infty)$ ,  $\psi(t, \varepsilon)$  is a continuously differentiable function with respect to  $t$  for  $(t, \varepsilon) \in R^+ \times (0, \varepsilon_1]$  into  $S_c$  and  $g$  is a continuous  $R^n$ -valued function defined on  $R^+ \times R^n \times (0, \varepsilon_1]$ .

We denote by  $x(t, \tau, \gamma, \lambda, \varepsilon)$  the solution of (1) satisfying  $x(\tau) = \gamma$  for  $\tau \in R^+$ ,  $\gamma \in R^n$ ,  $\lambda \in S_c$  and  $\varepsilon \in (0, \varepsilon_1]$ . It is known [11] that the matrices

$$\frac{\partial x}{\partial \gamma}(t, \tau, \gamma, \lambda, \varepsilon) \quad \text{and} \quad \frac{\partial x}{\partial \lambda}(t, \tau, \gamma, \lambda, \varepsilon)$$

exist and satisfy the equations

$$\dot{y} = f_x(t, x(t, \tau, \gamma, \lambda, \varepsilon), \lambda, \varepsilon) y$$

and

$$\dot{y} = f_x(t, x(t, \tau, \gamma, \lambda, \varepsilon), \lambda, \varepsilon) y + f_\lambda(t, x(t, \tau, \gamma, \lambda, \varepsilon), \lambda, \varepsilon)$$

respectively, with

$$\frac{\partial x}{\partial \gamma}(\tau, \tau, \gamma, \lambda, \varepsilon) = E \quad (E = \text{identity matrix})$$

and

$$\frac{\partial x}{\partial \lambda}(\tau, \tau, \gamma, \lambda, \varepsilon) = 0.$$

Furthermore

$$\frac{\partial x}{\partial \tau}(t, \tau, \gamma, \lambda, \varepsilon) = -\frac{\partial x}{\partial \gamma}(t, \tau, \gamma, \lambda, \varepsilon) f(\tau, \gamma, \lambda, \varepsilon).$$

for every

$$(\tau, \gamma, \lambda, \varepsilon) \in R^+ \times R^n \times S_c \times (0, \varepsilon_1].$$

The symbol  $|\cdot|$  will be used to denote any convenient vector norm in  $R^n$  and  $R^m$ .

Theorem (1) establishes asymptotic equivalence for the systems

$$\dot{x} = f(t, x, \psi(\tau, \varepsilon), \varepsilon) \tag{3}$$

and

$$\dot{y} = f(t, y, \psi(t, \varepsilon), \varepsilon) + g(t, y, \varepsilon) \tag{4}$$

for any fixed  $\tau \in R^+$  and for sufficiently small  $\varepsilon$ . In Theorem (2) we investigate this problem for differential equations of the form

$$\dot{x} = f(t, x, \lambda^*(\varepsilon), \varepsilon) \quad (5)$$

and (4), where  $\lambda^*(\varepsilon) = \lim_{t \rightarrow \infty} \psi(t, \varepsilon)$ . An example is given at the end of each theorem to illustrate the application of our results.

The following lemmas are needed for the proof of Theorem (1). Lemma 1 is a modified form of the Alekseev formula [1], which is due to Proctor [23]. Lemma 2 is analogous to Lemma 2 of [6].

LEMMA 1. *Let  $y(t)$  be a solution of (4) for  $\tau \leq t \leq \tau + \delta$  ( $\delta > 0$ ). Then  $y(t)$  satisfies*

$$y(t) = x(t, \tau, y(\tau), \psi(\tau, \varepsilon), \varepsilon) + \int_{\tau}^t H(t, s, y(s), \varepsilon) ds \quad (6)$$

where

$$\begin{aligned} H(t, s, y, \varepsilon) &= \frac{\partial x}{\partial y}(t, s, y, \psi(s, \varepsilon), \varepsilon) g(s, y, \varepsilon) \\ &\quad + \frac{\partial x}{\partial \lambda}(t, s, y, \psi(s, \varepsilon), \varepsilon) \dot{\psi}(s, \varepsilon), \quad 0 \leq s \leq t \leq \tau \end{aligned}$$

conversely if  $y$  is a solution of (6) for  $\tau \leq t \leq \tau + \delta$  then  $y$  satisfies (4).

LEMMA 2. *Let  $\gamma_1, \gamma_2 \in R^n$ ; for every  $t, \lambda$  and  $\varepsilon$  we have*

$$\begin{aligned} &|x(t, \tau, \gamma_1, \lambda, \varepsilon) - x(t, \tau, \gamma_2, \lambda, \varepsilon)| \\ &\leq |\gamma_1 - \gamma_2| \max_{0 \leq \theta \leq 1} \left| \frac{\partial x}{\partial \lambda}(t, \tau, \xi(\theta), \lambda, \varepsilon) \right| \end{aligned}$$

where  $\xi(\theta) = \gamma_1 + \theta(\gamma_2 - \gamma_1)$ ,  $0 \leq \theta \leq 1$ .

THEOREM 1. *Let  $(\tau, \gamma) \in R^+ \times R^n$  be fixed. Assume that there exists a bounded solution  $x(t) = x(t, \tau, \gamma, \psi(\tau, \varepsilon), \varepsilon)$  of (3) defined on  $[\tau, \infty)$  with bound independent of  $\varepsilon$ . Suppose that there are two functions  $w_1(t, s, r, \varepsilon)$  and  $w_2(t, s, r, \varepsilon)$ , both nondecreasing in  $r$  and both in the class  $c[[\tau, \infty) \times [\tau, \infty) \times [0, \infty) \times (0, \varepsilon_1], [0, \infty)]$  such that for every  $t, s, y$  and  $\varepsilon > 0$  we have*

- (i)  $|(\partial x / \partial y)(t, s, y, \psi(s, \varepsilon), \varepsilon)| \leq w_1(t, s, |y|, \varepsilon);$
- (ii)  $|g(t, y, \varepsilon)| \leq \alpha(t, |y|, \varepsilon)$ ,  $\alpha(t, r, \varepsilon)$  is nondecreasing in  $r$  and  $\alpha \in c[[\tau, \infty) \times [0, \infty) \times (0, \varepsilon_1], [0, \infty)];$

$$(iii) \quad |(\partial x / \partial \lambda)(t, s, y, \psi(s, \varepsilon), \varepsilon) \dot{\psi}(s, \varepsilon)| \leq w_2(t, s, |y|, \varepsilon);$$

(iv)  $w_1(t, s, a, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,  $w_2(t, s, a, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $w_1(t, s, a, \varepsilon) \alpha(s, a, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  for every fixed  $t$  and  $s$  and every  $a \geq 0$ , and also  $w_1(t, s, a, \varepsilon) \rightarrow 0$  as  $t \rightarrow \infty$ , for every fixed  $s$ ,  $a \geq 0$  and  $\varepsilon > 0$ ;

(v)  $\int_{\tau}^t w_1(t, s, a, \varepsilon) \alpha(s, a, \varepsilon) ds \rightarrow 0$  as  $t \rightarrow \infty$  and  $\int_{\tau}^t w_2(t, s, a, \varepsilon) ds \rightarrow 0$  as  $t \rightarrow \infty$  for every  $a \geq 0$  and every  $\varepsilon \in (0, \varepsilon_1]$ .

Then there is an  $\varepsilon_0$  with  $0 < \varepsilon_0 \leq \varepsilon_1$  such that for each  $0 < \varepsilon \leq \varepsilon_0$  there exists a bounded solution  $y(t) = y(t, \varepsilon)$  of (4) defined on  $[\tau, \infty)$  and satisfying

$$|x(t) - y(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

*Proof of Theorem 1.* For each bounded and continuous function  $y(t)$  defined on  $[\tau, \infty)$  with values in  $R^n$  we define the norm  $\|y\| = \sup_{t \geq \tau} |y(t)|$ . Since  $x(t)$  is bounded there is a constant  $\rho > 0$  independent of  $\varepsilon$  such that  $\|x\| \leq \rho/2$ . Consider the class of functions

$$B_\rho = \{U \in C[\tau, \infty), R^n \mid \|U\| \leq \rho\}.$$

Obviously  $B_\rho$  is a closed subset of the Banach space  $S$  of all bounded continuous  $R^n$ -valued functions defined on  $[\tau, \infty)$  with the uniform topology on compact intervals.

Let  $y \in B_\rho$ , and consider the operator  $\Pi$  defined by

$$(\Pi y)(t) = x(t, \tau, y(\tau), \psi(\tau, \varepsilon), \varepsilon) + \int_{\tau}^t H(t, s, y(s), \varepsilon) ds.$$

In order to prove that  $\Pi$  has a fixed point via the Schauder–Tychonoff fixed-point theorem we have to prove that [see 9]

(i)'  $\Pi: B_\rho \rightarrow B_\rho$  is continuous in the sense that if  $y_n \in B_\rho$  ( $n = 1, 2, 3, \dots$ ) and  $y_n \rightarrow y$  uniformly on every compact subinterval of  $[\tau, \infty)$ , then  $\Pi y_n \rightarrow \Pi y$  uniformly on every compact subinterval of  $[\tau, \infty)$ ;

(ii)' the functions in the image set  $\Pi B_\rho$  are uniformly bounded and equicontinuous at every point of  $[\tau, \infty)$ . From conditions (i)–(iii) and by Lemma (2) we obtain

$$\begin{aligned} |\Pi y(t)| &\leq \rho/2 + |y - y(\tau)| \max_{0 \leq \theta \leq 1} \left| \frac{\partial x}{\partial y}(t, \tau, \xi(\theta), \psi(\tau, \varepsilon), \varepsilon) \right| \\ &\quad + \sup_{t \in [\tau, \infty)} \left\{ \int_{\tau}^t [w_1(t, s, \rho, \varepsilon) \alpha(s, \rho, \varepsilon) + w_2(t, s, \rho, \varepsilon)] ds \right\}. \end{aligned}$$

Thus

$$|\Pi y(t)| \leq \rho/2 + |\gamma - y(\tau)| w_1(t, s, k, \varepsilon) \\ + \sup_{t \in [\tau, \infty)} \left\{ \int_{\tau}^t |w_1(t, s, \rho, \varepsilon) \alpha(s, \rho, \varepsilon) + w_2(t, s, \rho, \varepsilon)| ds \right\},$$

where  $k \geq |\gamma|(1 - \theta) + \theta\rho$  for every  $0 \leq \theta \leq 1$ . Now choose  $\varepsilon_0$  so small such that for  $0 < \varepsilon \leq \varepsilon_0$  the sum of the last two parts is less than  $\rho/2$ ; this is possible by (iv) and (v). So  $|\Pi y(t)| \leq \rho$ , i.e.,  $\Pi B \subset B_\rho$ .

We now prove that  $\Pi$  is continuous. Let  $y_n \in B_\rho$  and  $y_n \rightarrow y$  uniformly on every compact subset of  $[\tau, \infty)$ ; then it follows from the continuity of  $x$ ,  $\partial x / \partial \gamma$ ,  $\partial x / \partial \lambda$  and  $g$  with respect to initial values (also Lemma 2 and mean value theorem) that

$$x(t, \tau, y_n(\tau), \psi(\tau, \varepsilon), \varepsilon) \rightarrow x(t, \tau, y(\tau), \psi(\tau, \varepsilon), \varepsilon), \\ \frac{\partial x}{\partial \gamma}(t, s, y_n(s), \psi(s, \varepsilon), \varepsilon) \rightarrow \frac{\partial x}{\partial \gamma}(t, s, y(s), \psi(s, \varepsilon), \varepsilon), \\ \frac{\partial x}{\partial \lambda}(t, s, y_n(s), \psi(s, \varepsilon), \varepsilon) \rightarrow \frac{\partial x}{\partial \lambda}(t, s, y(s), \psi(s, \varepsilon), \varepsilon).$$

and

$$g(t, y_n(t), \varepsilon) \rightarrow g(t, y(t), \varepsilon)$$

as  $n \rightarrow \infty$  uniformly in  $t$  on every compact subinterval of  $[\tau, \infty)$ . So

$$\lim_{n \rightarrow \infty} \|\Pi y_n - \Pi y\| = 0,$$

whence  $\Pi$  is continuous.

To prove (ii)', it is enough to prove that the family  $\Pi_{B_\rho}$  is equicontinuous on any compact interval  $[a, b]$ ,  $b \geq a \geq \tau$ . That is, we have to prove that for each  $\varepsilon' > 0$  there is a  $\delta(\varepsilon') > 0$  such that  $\|\Pi y(t_1) - \Pi y(t_2)\| < \varepsilon'$  whenever  $|t_1 - t_2| < \delta(\varepsilon')$  for any  $a \leq t_1 \leq t_2 \leq b$  and  $y \in B_\rho$ .

Since  $x(t) = x(t, \tau, y(\tau), \psi(\tau, \varepsilon), \varepsilon)$  is continuous on  $[a, b]$ . Therefore given  $\varepsilon' > 0$  there is a  $\delta_1(\varepsilon') > 0$  such that  $|x(t_1) - x(t_2)| < \varepsilon'/2$  whenever  $|t_1 - t_2| < \delta_1(\varepsilon')$  for  $a \leq t_1 \leq t_2 \leq b$ . So if  $y \in B_\rho$  and  $|t_1 - t_2| < \delta_1(\varepsilon')$ , we obtain

$$|\Pi y(t_1) - \Pi y(t_2)| \\ \leq \varepsilon'/2 + \int_{t_1}^{t_2} \left| \frac{\partial x}{\partial \gamma}(t_2, s, y(s), \psi(s, \varepsilon), \varepsilon) g(s, y(s), \varepsilon) \right| ds \\ + \int_{\tau}^{t_1} \left| \frac{\partial x}{\partial \gamma}(t_1, s, y(s), \psi(s, \varepsilon), \varepsilon) - \frac{\partial x}{\partial \gamma}(t_2, s, y(s), \psi(s, \varepsilon), \varepsilon) \right| \\ \times |g(s, y(s), \varepsilon)| ds$$

$$\begin{aligned}
& + \int_{\tau}^t \left| \frac{\partial x}{\partial \lambda} (t_1, s, y(s), \psi(s, \varepsilon), \varepsilon) - \frac{\partial x}{\partial \lambda} (t_2, s, y(s), \psi(s, \varepsilon), \varepsilon) \right| \\
& \times |\dot{\psi}(s, \varepsilon)| ds \\
& + \int_{t_1}^{t_2} \left| \frac{\partial x}{\partial \lambda} (t_2, s, y(s), \psi(s, \varepsilon), \varepsilon) \dot{\psi}(s, \varepsilon) \right| ds.
\end{aligned}$$

Thus from conditions (i)–(iii) we have

$$\begin{aligned}
& |\Pi y(t_1) - \Pi y(t_2)| \\
& \leq \varepsilon'/2 + \int_{t_1}^{t_2} [w_1(t_2, s, \rho, \varepsilon) \alpha(s, \rho, \varepsilon) + w_2(t_2, s, \rho, \varepsilon)] ds \\
& + \int_{\tau}^{t_1} \left| \frac{\partial x}{\partial \gamma} (t_1, s, y(s), \psi(s, \varepsilon), \varepsilon) - \frac{\partial x}{\partial \gamma} (t_2, s, y(s), \psi(s, \varepsilon), \varepsilon) \right| \\
& \times |g(s, y(s), \varepsilon)| ds \\
& \pm \int_{\tau}^{t_1} \left| \frac{\partial x}{\partial \lambda} (t_1, s, y(s), \psi(s, \varepsilon), \varepsilon) - \frac{\partial x}{\partial \lambda} (t_2, s, y(s), \psi(s, \varepsilon), \varepsilon) \right| \\
& \times |\dot{\psi}(s, \varepsilon)| ds.
\end{aligned}$$

By the mean value theorem there exists  $\xi \in (t_1, t_2)$  such that

$$\begin{aligned}
& |\Pi y(t_1) - \Pi y(t_2)| \\
& \leq \varepsilon'/2 + |t_1 - t_2| \{w_1(t_2, \xi, \rho, \varepsilon) \alpha(\xi, \rho, \varepsilon) + w_2(t_2, \xi, \rho, \varepsilon)\} \\
& + \int_{\tau}^{t_1} \left| \frac{\partial x}{\partial \gamma} (t_1, s, y(s), \psi(s, \varepsilon), \varepsilon) - \frac{\partial x}{\partial \gamma} (t_2, s, y(s), \psi(s, \varepsilon), \varepsilon) \right| \\
& \times |g(s, y(s), \varepsilon)| ds \\
& + \int_{\tau}^{t_1} \left| \frac{\partial x}{\partial \lambda} (t_1, s, y(s), \psi(s, \varepsilon), \varepsilon) - \frac{\partial x}{\partial \lambda} (t_2, s, y(s), \psi(s, \varepsilon), \varepsilon) \right| \\
& |\dot{\psi}(s, \varepsilon)| ds.
\end{aligned}$$

By applying the mean value theorem, these last two integrals are dominated by the quantities

$$\begin{aligned}
& |t_1 - t_2| \int_{\tau}^{t_1} \left| \{f_x(\sigma_1(s), x(\sigma_1(s), s, y(s), \psi(s, \varepsilon), \varepsilon), \psi(s, \varepsilon), \varepsilon)\} \right. \\
& \times \left. \left\{ \frac{\partial x}{\partial \gamma} (\sigma_1(s), s, y(s), \psi(s, \varepsilon), \varepsilon) \right\} \right| |g(s, y(s), \varepsilon)| ds
\end{aligned}$$

and

$$\begin{aligned}
 & |t_1 - t_2| \int_{\tau}^{t_1} \left| \{f_x(\sigma_2(s), x(\sigma_2(s), s, y(s), \psi(s, \varepsilon), \varepsilon), \psi(s, \varepsilon), \varepsilon)\} \right. \\
 & \quad \times \left. \left\{ \frac{\partial x}{\partial \lambda}(\sigma_2(s), s, y(s), \psi(s, \varepsilon), \varepsilon) \right\} \right. \\
 & \quad \left. + f_{\lambda}(\sigma_2(s), x(\sigma_2(s), s, y(s), \psi(s, \varepsilon), \varepsilon), \psi(s, \varepsilon), \varepsilon) \right| |\dot{\psi}(s, \varepsilon)| ds,
 \end{aligned}$$

respectively, where  $t_1 \leq \sigma_1(s) \leq t_2$ ,  $t_1 \leq \sigma_2(s) \leq t_2$  and  $\tau \leq s \leq t_1$ . Moreover, there exists a constant independent of  $t_1$ ,  $t_2$ ,  $\xi$  and  $y$  which in turn dominates each of the above integrands so long as  $a \leq t_1 \leq t_2 \leq b$ ,  $y \in B_\rho$  and  $|t_1 - t_2| < \delta_1(\varepsilon')$ . Hence there exists a positive number  $\delta(\varepsilon') \leq \delta_1(\varepsilon')$  such that

$$|\Pi y(t_1) - \Pi y(t_2)| < \varepsilon' \quad \text{whenever} \quad |t_1 - t_2| < \delta(\varepsilon').$$

This establishes the condition (ii)'. Thus  $\Pi y = y$  has a solution  $y$  on  $[\tau, \infty)$  and by Lemma 1 this solution satisfies (4). We have

$$\begin{aligned}
 |y(t) - x(t)| & \leq |x(t, \tau, y(\tau), \psi(\tau, \varepsilon), \varepsilon) - x(t)| \\
 & \quad + \int_{\tau}^t [w_1(t, s, |y|, \varepsilon) \alpha(s, |y|, \varepsilon) + w_2(t, s, |y|, \varepsilon)] ds.
 \end{aligned}$$

Since the first term is bounded by  $|x - y(\tau)| w_1(t, \tau, |y|, \varepsilon)$ , the entire quantity tends to 0 as  $t \rightarrow \infty$ . This completes the proof of Theorem 1.

For constructing an example, the following lemma is useful.

LEMMA 3 (see [10]). *Let  $T \geq 0$  and  $v > 0$ , and suppose that  $\alpha(t)$  is a continuous function for  $t \geq T$ , nonpositive and bounded below, with*

$$\overline{\lim}_{t \rightarrow \infty} \int_t^{t+v} \alpha(u) du = \sigma < 0, \quad \text{for some } v > 0,$$

*and that  $\beta(t)$  is a continuous nonnegative function for  $t \geq T$  such that*

$$\lim_{t \rightarrow \infty} \int_t^{t+\delta} \beta(u) du = 0$$

*for some  $\delta > 0$ . Then we have*

$$\lim_{t \rightarrow \infty} \int_{\tau}^t \exp \left( \int_{\tau}^t \alpha(u) du \right) \beta(s) ds = 0$$

and also

$$\lim_{t \rightarrow \infty} \exp \left( \int_{\tau}^t \alpha(u) du \right) = 0.$$

EXAMPLE. Let  $h(t, y)$  be a continuous function on  $[0, \infty) \times R^n$  such that  $|h(t, y)| \leq M |y| \beta(t)$ , where  $M$  is a nonnegative continuous function and  $\beta(t)$  is a continuous nonnegative function for  $t \geq 0$  such that

$$\lim_{t \rightarrow \infty} \int_t^{t+\delta} \beta(u) du = 0, \quad \text{for some } \delta > 0.$$

We prove that for sufficiently small  $\varepsilon$ , the differential equation

$$\dot{y} + \frac{k}{\varepsilon} y = h(t, y), \quad k > 0,$$

has a solution  $y(t)$  defined on  $[0, \infty)$ , such that  $|y(t)| \rightarrow 0$  as  $t \rightarrow \infty$ . If we take  $\psi(t, \varepsilon) = k$  and  $g(t, y, \varepsilon) = h(t, y)$ , then the corresponding parametric unperturbed equation would be

$$\dot{x} + \frac{\lambda}{\varepsilon} x = 0, \quad \lambda > 0, \quad \varepsilon > 0.$$

The equations corresponding to (3) and (4) are

$$\dot{x} + (k/\varepsilon) x = 0, \tag{7}$$

$$\dot{y} + (k/\varepsilon) y = h(t, y), \tag{8}$$

respectively. Also

$$x(t, \tau, \gamma, \lambda, \varepsilon) = \gamma e^{-\lambda/\varepsilon(t-\tau)},$$

$$\left| \frac{\partial x}{\partial \gamma}(t, s, y, \psi(s, \varepsilon), \varepsilon) \right| = e^{-k/\varepsilon(t-s)} = w_1(t, s, |y|, \varepsilon),$$

$$\left| \frac{\partial x}{\partial \lambda}(t, s, y, \psi(s, \varepsilon), \varepsilon) \right| \leq \frac{|y|}{\varepsilon} (t-s) e^{-k/\varepsilon(t-s)} = w_2(t, s, |y|, \varepsilon),$$

$$|h(t, y)| \leq M(|y|) \beta(t) = \alpha(t, |y|, \varepsilon),$$

$$\int_0^t w_1(t, s, a, \varepsilon) \alpha(s, a, \varepsilon) ds \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (\text{by Lemma 3}).$$

All conditions of Theorem 1 are satisfied. Since  $x(t) \equiv 0$  is a bounded solution of (7), so there exists a solution  $y(t)$  defined on  $[0, \infty)$  such that  $|y(t)| \rightarrow 0$  as  $t \rightarrow \infty$ , for sufficiently small  $\varepsilon$ .



**THEOREM 2.** Assume that there exists a bounded solution  $x(t) = x(t, \tau, \gamma, \lambda^*, \varepsilon) = x(t, \lambda^*(\varepsilon))$  of (5) defined on  $[\tau, \infty)$ , with bound independent of  $\varepsilon$ . Suppose there are functions  $\alpha_i(t, s, \varepsilon)$  in  $C[[\tau, \infty) \times [\tau, \infty) \times (0, \varepsilon_1], [0, \infty)]$ ,  $i = 1, 2, 3, 4$ , with properties

$$(i) \quad |(\partial x / \partial \gamma)(t, s, y, \psi(s, \varepsilon), \varepsilon) g(s, y, \varepsilon) - (\partial x / \partial \gamma)(t, s, z, \psi(s, \varepsilon), \varepsilon) g(s, z, \varepsilon)| \leq \alpha_1(t, s, \varepsilon) |y - z|,$$

$$(ii) \quad |(\partial x / \partial \gamma)(t, s, 0, \psi(s, \varepsilon), \varepsilon) g(s, 0, \varepsilon)| \leq \alpha_2(t, s, \varepsilon),$$

$$(iii) \quad |((\partial x / \partial \lambda)(t, s, y, \psi(s, \varepsilon), \varepsilon) - (\partial x / \partial \lambda)(t, s, z, \psi(s, \varepsilon), \varepsilon)) \dot{\psi}(s, \varepsilon)| \leq \alpha_3(t, s, \varepsilon) |y - z|,$$

$$(iv) \quad |(\partial x / \partial \lambda)(t, s, 0, \psi(s, \varepsilon), \varepsilon) \dot{\psi}(s, \varepsilon)| \leq \alpha_4(t, s, \varepsilon),$$

$$(v) \quad \int_{\tau}^{\infty} \alpha_i(T, s, \varepsilon) ds \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \text{ for each } i = 1, 2, 3, 4,$$

(vi)  $M_i(T, \varepsilon) = \sup\{\int_{\tau}^{\infty} \alpha_i(s, u, \varepsilon) du \mid \tau \leq s \leq T\} \rightarrow 0$  as  $T \rightarrow \infty$ , for each  $\varepsilon \in (0, \varepsilon_1]$ ,  $i = 1, 2, 3, 4$ , and  $\sup\{|x(s, T, x(T, \lambda^*(\varepsilon)), \psi(T, \varepsilon), \varepsilon) - x(s, \lambda^*(\varepsilon))| \mid \tau \leq s \leq T\} \rightarrow 0$  as  $T \rightarrow \infty$  for each  $\varepsilon \in (0, \varepsilon_1]$ . Then there is an  $\varepsilon_0 > 0$  such that for each  $0 < \varepsilon < \varepsilon_0$  there exists a bounded solution  $y(t)$  of (4) defined on  $[\tau, \infty)$  satisfying  $|x(t) - y(t)| \rightarrow 0$  as  $t \rightarrow \infty$ .

*Proof.* Let  $\rho > 0$  be such that  $\|x\| \leq \rho/3$ . Let us define on  $B_{\rho}$  an operator  $\Pi$  as follows:

$$\begin{aligned} (\Pi y)(t) &= x(t) - \int_t^{\infty} \frac{\partial x}{\partial \gamma}(t, s, y(s), \psi(s, \varepsilon), \varepsilon) g(s, y(s), \varepsilon) ds \\ &\quad - \int_t^{\infty} \frac{\partial x}{\partial \lambda}(t, s, y(s), \psi(s, \varepsilon), \varepsilon) \dot{\psi}(s, \varepsilon) ds \\ &= x(t) - \int_t^{\infty} H(t, s, y(s), \varepsilon) ds. \end{aligned} \quad (9)$$

It is clear that  $\Pi$  is well defined. Also

$$\begin{aligned} |\Pi y(t)| &\leq \rho/3 + \rho \left\{ \int_t^{\infty} \alpha_1(t, s, \varepsilon) ds + \int_t^{\infty} \alpha_3(t, s, \varepsilon) ds \right. \\ &\quad \left. + \int_t^{\infty} \alpha_2(t, s, \varepsilon) ds + \int_t^{\infty} \alpha_4(t, s, \varepsilon) ds \right\}. \end{aligned}$$

Now choose  $\varepsilon_0 > 0$  such that for  $0 < \varepsilon \leq \varepsilon_0$  we have

$$\left\{ \int_t^{\infty} [\alpha_1(t, s, \varepsilon) + \alpha_3(t, s, \varepsilon)] ds \right\} \leq \frac{1}{3}$$

and

$$\left\{ \int_t^{\infty} |\alpha_2(t, s, \varepsilon) + \alpha_3(t, s, \varepsilon)| ds \right\} \leq \frac{\rho}{3}.$$

Thus  $\Pi B_\rho \subset B_\rho$ . Also

$$\begin{aligned} \|\Pi y(t) - \Pi z(t)\| &\leq \left( \int_t^{\infty} \alpha_1(t, s, \varepsilon) ds \right) \|y - z\| + \left( \int_t^{\infty} \alpha_3(t, s, \varepsilon) ds \right) \|y - z\| \\ &\leq \frac{1}{3} \|y - z\| \end{aligned}$$

for every  $y, z \in B_\rho$ . So  $\Pi$  is a contraction mapping, and by the contraction mapping principle there is a unique solution of  $\Pi y = y$ .

We next show that this solution  $y(t)$  satisfies (4). If we try to differentiate the equation  $y = \Pi y$ , in general the conditions of the contraction mapping theorem do not guarantee that  $\dot{y}(t)$  exists. So we prove that  $y$  is a uniform limit of certain solutions  $y_T$  of (4). In fact,  $y_T$  is the solution on  $\tau \leq t \leq T$  of the (proper) integral equation

$$y_T(t) = x(t, T, y_T(T), \psi(T, \varepsilon), \varepsilon) - \int_t^T H(t, s, y_T(s), \varepsilon) ds, \quad (10)$$

and can be obtained in a similar way as the unique fixed-point of an appropriate contraction mapping; here

$$x(t, T, y_T(T), \psi(T, \varepsilon), \varepsilon)$$

is a bounded solution of

$$\dot{x} = f(t, x, \psi(T, \varepsilon), \varepsilon), \quad \text{where } y_T(T) = x(T, \lambda^*(\varepsilon)).$$

We prove that  $y_T$  satisfies (4) on  $\tau \leq t \leq T$ . For simplicity, in the sequel we denote  $f_x$  and  $f_\lambda$  the values of these functions at  $(t, x(t, s, y_T(s), \psi(s, \varepsilon), \varepsilon), \psi(s, \varepsilon), \varepsilon)$  and we evaluate  $\partial x / \partial \gamma$  and  $\partial x / \partial \lambda$  at  $(t, s, y_T(s), \psi(s, \varepsilon), \varepsilon)$ . By differentiating  $y_T$  with respect to  $t$  we obtain

$$\begin{aligned} \dot{y}_T(t) &= f(t, x(t, T, y_T(T), \psi(T, \varepsilon), \varepsilon), \psi(T, \varepsilon), \varepsilon) + g(t, y_T(t), \varepsilon) \\ &\quad - \int_t^T f_x \frac{\partial x}{\partial \gamma} g(s, y_T(s), \varepsilon) ds - \int_t^T f_x \frac{\partial x}{\partial \lambda} \dot{\psi}(s, \varepsilon) ds \\ &\quad - \int_t^T f_\lambda \dot{\psi}(s, \varepsilon) ds. \end{aligned}$$

Also,

$$\begin{aligned} & \int_t^T \frac{df}{ds} (t, x(t, s, y_T(s), \psi(s, \varepsilon), \varepsilon), \psi(s, \varepsilon), \varepsilon) ds \\ &= f(t, x(t, T, y_T(T), \psi(T, \varepsilon), \varepsilon), \psi(T, \varepsilon), \varepsilon) - f(t, y_T(t), \psi(t, \varepsilon), \varepsilon). \end{aligned}$$

On the other hand, since

$$\begin{aligned} & \frac{df}{ds} (t, x(t, s, y_T(s), \psi(s, \varepsilon), \varepsilon), \psi(s, \varepsilon), \varepsilon) \\ &= f_x \frac{dx}{ds} (t, s, y_T(s), \psi(s, \varepsilon), \varepsilon) + f_\lambda \dot{\psi}(s, \varepsilon) \end{aligned}$$

and

$$\begin{aligned} & \frac{dx}{ds} (t, s, y_T(s), \psi(s, \varepsilon), \varepsilon) \\ &= \frac{\partial x}{\partial s} (t, s, y_T(s), \psi(s, \varepsilon), \varepsilon) + \frac{\partial x}{\partial \gamma} \dot{y}_T(s) + \frac{\partial x}{\partial \lambda} \dot{\psi}(s, \varepsilon), \end{aligned}$$

we have

$$\begin{aligned} \dot{y}_T(t) &= f(t, y_T(t), \psi(t, \varepsilon), \varepsilon) + g(t, y_T(t), \varepsilon) \\ &\quad - \int_t^T f_x \frac{\partial x}{\partial \gamma} g(s, y_T(s), \varepsilon) ds \\ &\quad + \int_t^T f_x \frac{\partial x}{\partial \gamma} \dot{y}_T(s) ds \\ &\quad + \int_t^T f_x \frac{\partial x}{\partial s} (t, s, y_T(s), \psi(s, \varepsilon), \varepsilon) ds. \end{aligned} \tag{11}$$

Let  $w(t) = f(t, y_T(t), \psi(t, \varepsilon), \varepsilon) + g(t, y_T(t), \varepsilon) - \dot{y}_T(t)$ . From (11) we obtain

$$w(t) = \int_t^T f_x \frac{\partial x}{\partial \gamma} w(s) ds.$$

Using the Gronwall inequality it is easy to see that this integral equation admits only the trivial solution

$$w(t) \equiv 0 \quad \text{on} \quad \tau_0 \leq t \leq T.$$

Therefore  $y_T$  is a solution of (4).

From (10) and the equation  $y' = Hy$  we readily obtain

$$\begin{aligned}
 |y_T(t) - y(t)| &\leq |x(t, T, y_T(T), \psi(T, \varepsilon), \varepsilon) - x(t, \lambda^*(\varepsilon))| \\
 &\quad + \int_t^T \left| \frac{\partial x}{\partial y}(t, x, Y_T(s), \psi(s, \varepsilon), \varepsilon) g(s, Y_T(s), \varepsilon) \right. \\
 &\quad \left. - \frac{\partial x}{\partial y}(t, s, y(s), \psi(s, \varepsilon), \varepsilon) g(s, y(s), \varepsilon) \right| ds \\
 &\quad + \int_t^T \left| \frac{\partial x}{\partial \lambda}(t, s, y_T(s), \psi(s, \varepsilon), \varepsilon) \right. \\
 &\quad \left. - \frac{\partial x}{\partial \lambda}(t, s, y(s), \psi(s, \varepsilon), \varepsilon) \right\} \left| \dot{\psi}(s, \varepsilon) \right| ds \\
 &\quad + \int_T^\infty \left| \frac{\partial x}{\partial y}(t, s, y(s), \psi(s, \varepsilon), \varepsilon) g(s, y(s), \varepsilon) \right. \\
 &\quad \left. + \frac{\partial x}{\partial \lambda}(t, s, y(s), \psi(s, \varepsilon), \varepsilon) \dot{\psi}(s, \varepsilon) \right| ds.
 \end{aligned}$$

So

$$\begin{aligned}
 |y_T(t) - y(t)| &\leq |x(t, T, y_T(T), \psi(T, \varepsilon), \varepsilon) - x(t, \lambda^*(\varepsilon))| \\
 &\quad + \left( \int_t^T \alpha_1(t, s, \varepsilon) ds \right) \sup\{|y_T(s) - y(s)| \mid \tau \leq s \leq T\} \\
 &\quad + \left( \int_t^T \alpha_3(t, s, \varepsilon) ds \right) \sup\{|y_T(s) - y(s)| \mid \tau \leq s \leq T\} \\
 &\quad + \rho \int_T^\infty (\alpha_1(t, s, \varepsilon) + \alpha_3(t, s, \varepsilon)) ds \\
 &\quad + \int_T^\infty (\alpha_2(t, s, \varepsilon) + \alpha_4(t, s, \varepsilon)) ds,
 \end{aligned}$$

and upon using (v) we obtain

$$\begin{aligned}
 &\sup\{|y_T(s) - y(s)| \mid \tau \leq s \leq T\} \\
 &\leq \frac{3}{2} \rho (M_1(T, \varepsilon) + M_3(T, \varepsilon)) + \frac{3}{2} (M_2(T, \varepsilon) + M_4(T, \varepsilon)) \\
 &\leq + \frac{3}{2} \sup\{|x(s, T, x(T, \lambda^*(\varepsilon)), \psi(T, \varepsilon), \varepsilon) - x(s, \lambda^*(\varepsilon))| \mid \tau \leq s \leq T\}.
 \end{aligned}$$

Therefore

$$\sup\{|y_T(s) - y(s)| \mid \tau \leq s \leq T\} \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

This shows that the solution  $y$  of (9) is the uniform limit (on compact subsets of  $[\tau, \infty)$ ) of solutions  $y_\tau$  of (4). Thus, any solution  $y$  of (9) is also a solution of (4). Clearly  $|y(t) - x(t)| \rightarrow 0$  as  $t \rightarrow \infty$ , which completes the proof of Theorem 2.

EXAMPLE. We prove that for sufficiently small  $\varepsilon$ , the differential equation

$$\dot{y} - \varepsilon \exp(-t) y = \frac{\varepsilon}{(t+1)^2} (\sin y + 1) \quad (12)$$

has a solution  $y(t)$  defined on  $[0, \infty)$ , such that  $|y(t)| \rightarrow 0$  as  $t \rightarrow \infty$ . If we take  $\psi(t, \varepsilon) = \varepsilon \exp(-t)$  and  $g(t, y, \varepsilon) = (\varepsilon/(t+1)^2)(\sin y + 1)$ , then the corresponding parametric unperturbed system is

$$\dot{x} - \lambda x = 0, \quad \lambda \in [0, 1].$$

Since  $\lambda^*(\varepsilon) \equiv 0$ , the equations corresponding to (5) and (4) are

$$\dot{x} = 0 \quad (13)$$

and (12), respectively. Also

$$\begin{aligned} & \left| \frac{\partial x}{\partial \gamma}(t, s, y, \psi(s, \varepsilon), \varepsilon) g(s, y, \varepsilon) - \frac{\partial x}{\partial \gamma}(t, s, z, \psi(s, \varepsilon), \varepsilon) g(s, z, \varepsilon) \right| \\ & \leq |g(s, y, \varepsilon) - g(s, z, \varepsilon)| \\ & \leq \frac{\varepsilon}{(s+1)^2} |y - z|, \\ & \left| \frac{\partial x}{\partial \gamma}(t, s, 0, \psi(s, \varepsilon), \varepsilon) g(s, 0, \varepsilon) \right| \leq |g(s, 0, \varepsilon)| = \frac{\varepsilon}{(s+1)^2}, \\ & \left| \frac{\partial x}{\partial \lambda}(t, x, y, \psi(s, \varepsilon), \varepsilon) - \frac{\partial x}{\partial \lambda}(t, s, z, \psi(s, \varepsilon), \varepsilon) \right\} \psi(s, \varepsilon) \Big| \\ & \leq \varepsilon(s-t) \exp(-s) |y - z|, \quad 0 \leq t \leq s < \infty, \end{aligned}$$

so we take  $\alpha_1(t, s, \varepsilon) = \alpha_2(t, s, \varepsilon) = \varepsilon/(s+1)^2$ ,

$$\alpha_3(t, s, \varepsilon) = \varepsilon(s-t) \exp(-s), \quad \alpha_4(t, s, \varepsilon) \equiv 0.$$

The conditions of Theorem 2 are satisfied, since  $x \equiv 0$  is a bounded solution of (13); so there exists a bounded solution  $y(t)$  of (12) defined on  $[0, \infty)$ , such that  $|y(t)| \rightarrow 0$ , as  $t \rightarrow \infty$ , for sufficiently small  $\varepsilon > 0$ .

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